CMSC828T
Vision, Planning And Control In Aerial Robotics

RIGID BODY TRANSFORMATIONS

Rigid Body Transformations

Most of these slides are
inspired by MEAM620 Slides at UPenn

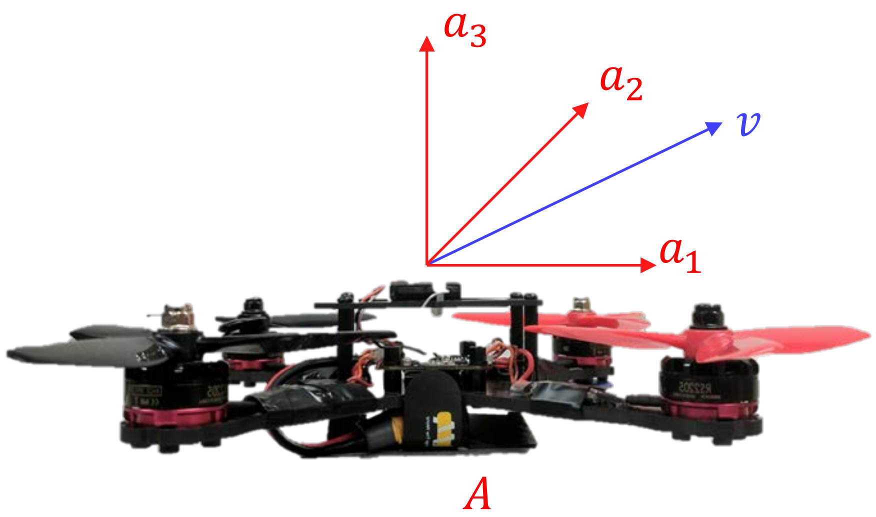
Reference Frames

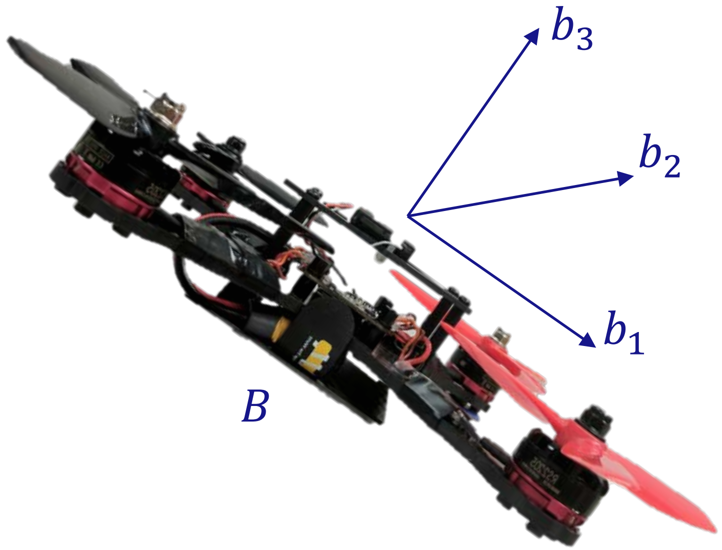
We associate an **orthonormal** reference frame to any position and orientation

In frame A we can find 3 linearly independent vectors $[a_1, a_2, a_3]^T$ that are basis vectors

We can write any vector in a frame as a linear combination of the basis vectors

$$v = v_1 a_1 + v_2 a_2 + v_3 a_3$$





Notations

v Vector (vertical) or its component

A Reference Frames

$\mathbf{A}$ Matrices

${}^{\mathbf{A}}\mathbf{A}_{\mathbf{B}}$ Transformations

Big Potential for confusion!

We'll try to stick to these norms,
if something is not obvious please ask

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Rigid Body

Rigid Body

What do mean by a rigid body?

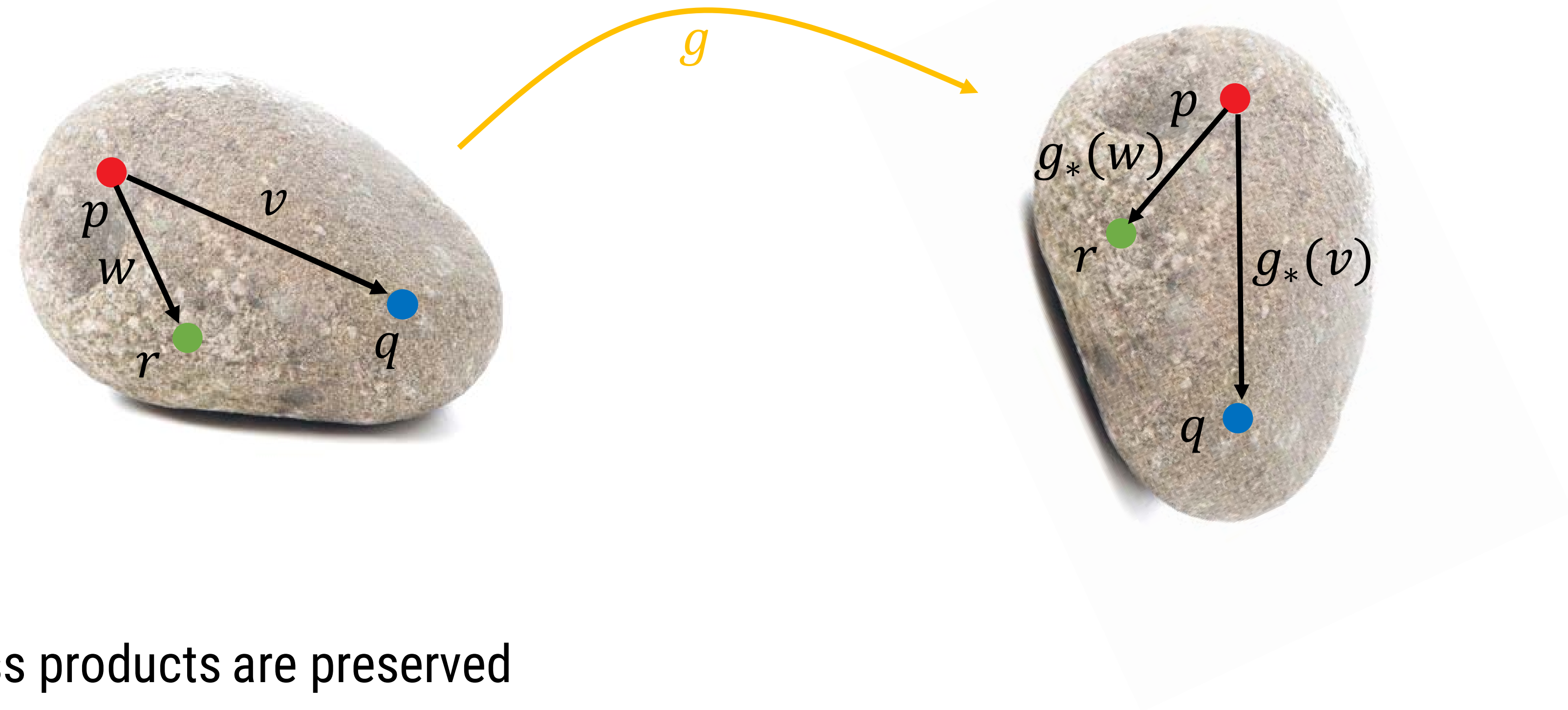
Rigid Body Displacement

A displacement is a transformation of points

1. Lengths are preserved

$$||g(p) - g(q)|| = ||p - q||$$

Rigid Body Displacement



2. Cross products are preserved

$$g_*(v) \times g_*(w) = g_*(v \times w)$$

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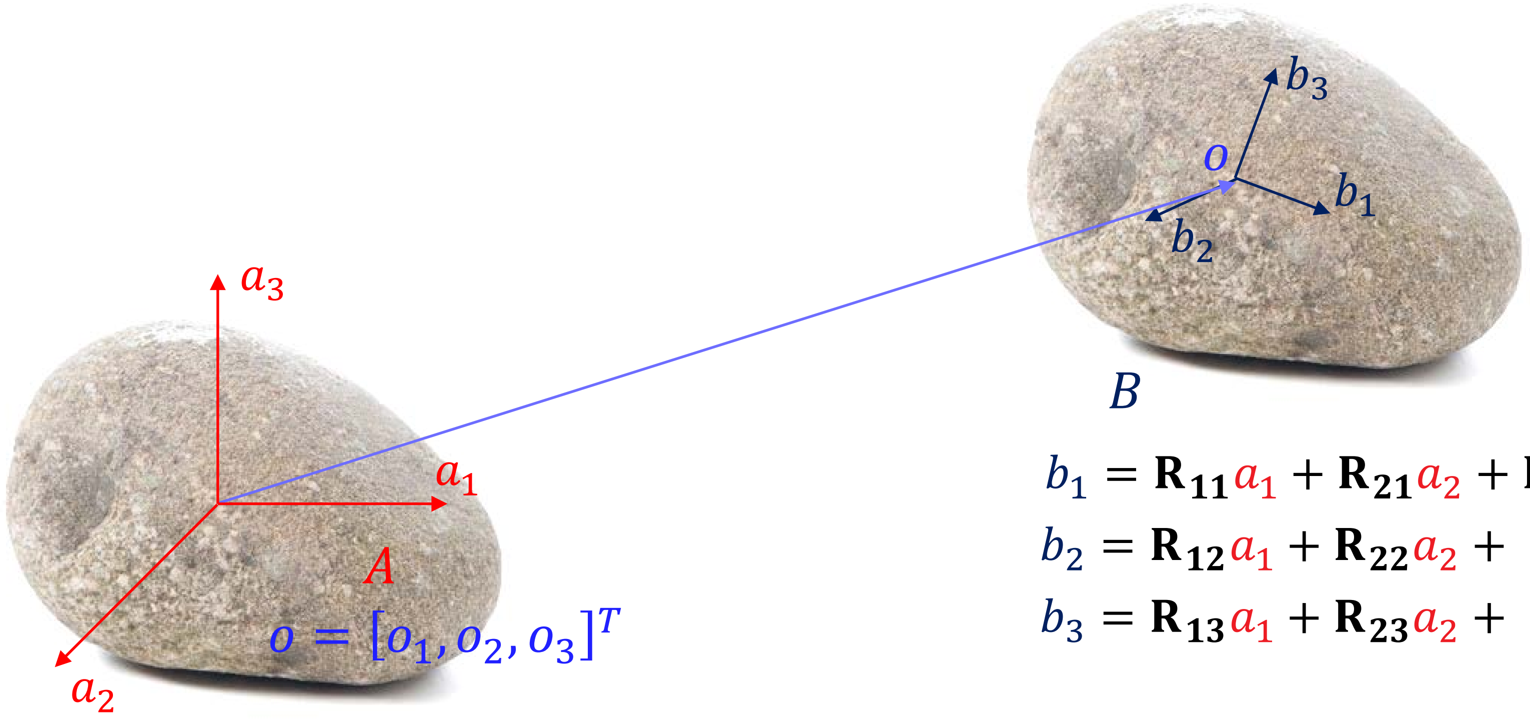
Points to remember

Rigid body displacements are transformations (maps) that satisfy these important properties

1. The map preserves lengths
2. Cross products are preserved by the induced map
3. Rigid body displacements and rigid body transformations are used interchangeably
4. Transformations generally used to describe relationship between reference frames attached to different rigid bodies
5. Displacements describe relationships between two positions and orientation of a frame attached to a displaced rigid body

Rigid Body Pose

Once chosen a frame, a rigid body is described in space by its position and orientation with respect to that reference frame



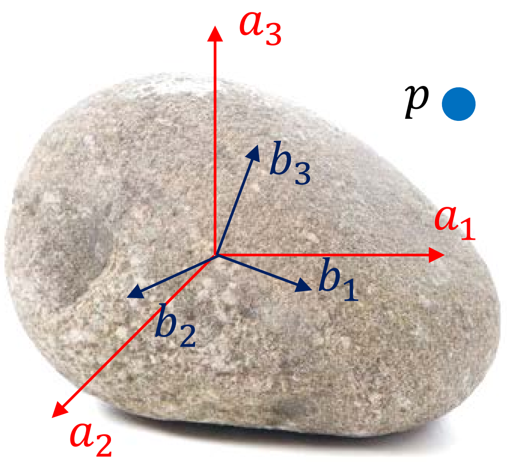
$$b_1 = \mathbf{R}_{11} a_1 + \mathbf{R}_{21} a_2 + \mathbf{R}_{31} a_3$$

$$b_2 = \mathbf{R}_{12} a_1 + \mathbf{R}_{22} a_2 + \mathbf{R}_{32} a_3$$

$$b_3 = \mathbf{R}_{13} a_1 + \mathbf{R}_{23} a_2 + \mathbf{R}_{33} a_3$$

Rotation Matrix

Consider only rotation for this case such that the origins of both the frames are aligned



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$$b_3 = \mathbf{R_{13}} a_1 + \mathbf{R_{23}} a_2 + \mathbf{R_{33}} a_3$$

$$\mathbf{R}_{\mathbf{B}}^{\mathbf{A}} = [b_1 b_2 b_3] = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} b_1^T a_1 & b_2^T a_1 & b_3^T a_1 \\ b_1^T a_2 & b_2^T a_2 & b_3^T a_2 \\ b_1^T a_3 & b_2^T a_3 & b_3^T a_3 \end{bmatrix}$$

$$p_A = p_1 b_1 + p_2 b_2 + p_3 b_3 = [b_1 b_2 b_3] p_B \qquad p_A = \mathbf{R}_{\boldsymbol{B}}^{\boldsymbol{A}} p_B \qquad \mathbf{R}_{\boldsymbol{B}}^{\boldsymbol{A}} = {}^{A}\mathbf{R}_{\boldsymbol{B}}$$

$\mathbf{R}_{\boldsymbol{B}}^{\boldsymbol{A}} = {}^{A}\mathbf{R}_{\boldsymbol{B}}$ takes points represented with respect to frame B and finds their locations with respect to frame A

Columns of $\mathbf{R}_{\boldsymbol{B}}^{\boldsymbol{A}}$ are the basis vectors for B represented in A

Rotation Matrix Properties

Orthogonal: $\mathbf{R}\mathbf{R}^T = \mathbf{I}$

Special Orthogonal: $\det \mathbf{R} = +1$

Closed under multiplication: $\mathbf{R}_1\mathbf{R}_2 = \mathbf{R}_3$

Inverse is a rotation matrix: $\mathbf{R}^{-1}$

Set of all rotation matrices is a group

What is a group?

$$G(A,\otimes)$$

Closure: $A, B \in G \Rightarrow A \otimes B \in G$

Associativity: $A, B, C \in G \Rightarrow (A \otimes B) \otimes C = A \otimes (B \otimes C)$

Identity Element: $A, I \in G \Rightarrow A \otimes I = I \otimes A = A$

Inverse Element: $A \in G, A \otimes B = B \otimes A = I \Rightarrow B \in G$

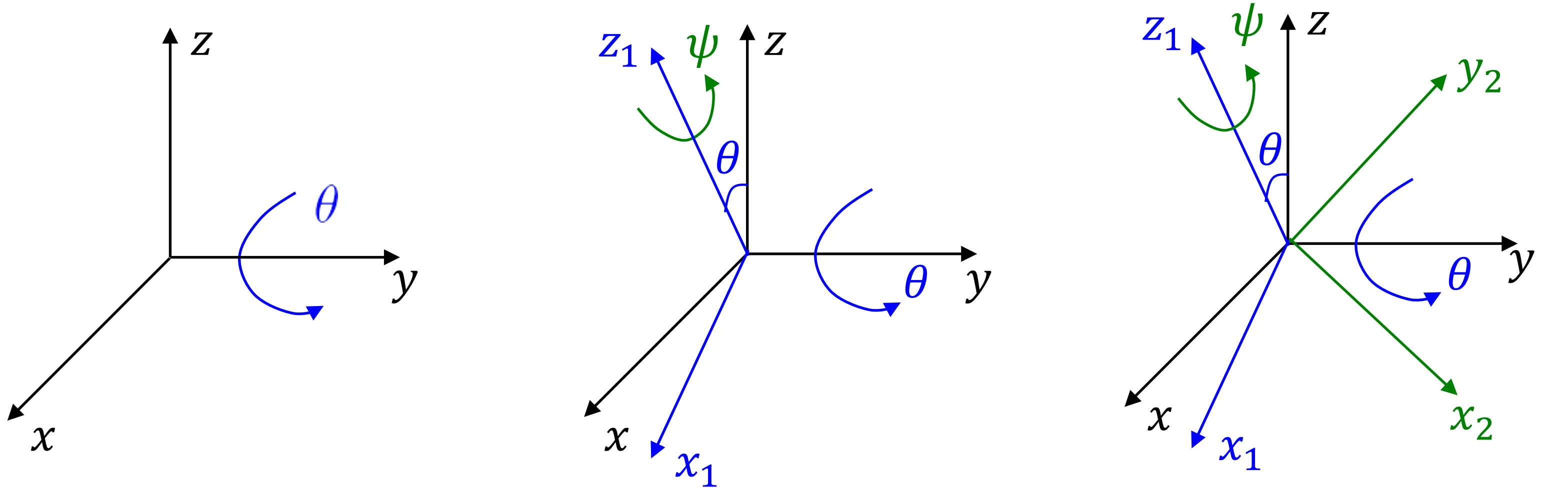
Why is set of rotation matrices a group?

$$G(\mathbf{R},\times) = \{\mathbf{R} \in G | \mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}, \det \mathbf{R} = +1\}$$

$$\mathbf{R} \in SO(3)$$

$$SO(3) = \{\mathbf{R} \in \mathbb{R}^{3\times3} | \mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}, \det \mathbf{R} = +1\}$$

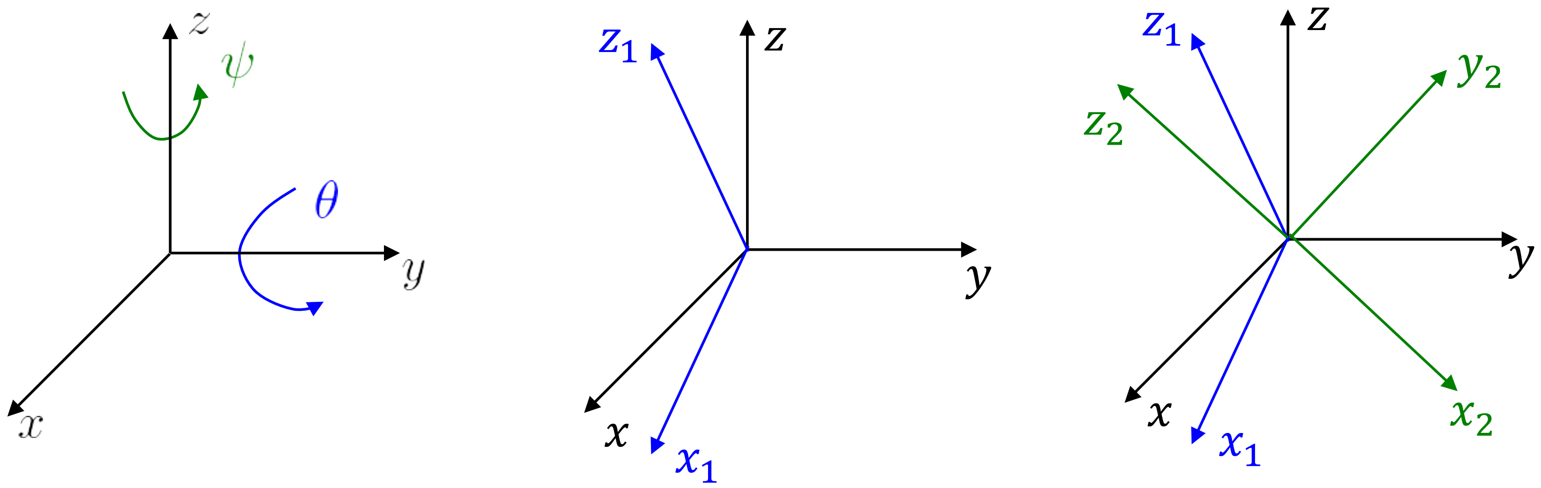
Rotations about intermediate axes



$$\mathbf{R}_2^0 = \mathbf{R}_1^0 \mathbf{R}_2^1 = \mathbf{R}_{y,\theta} \mathbf{R}_{z,\psi}$$

Post-multiply successive transformations about intermediate frames

Rotations about fixed axes



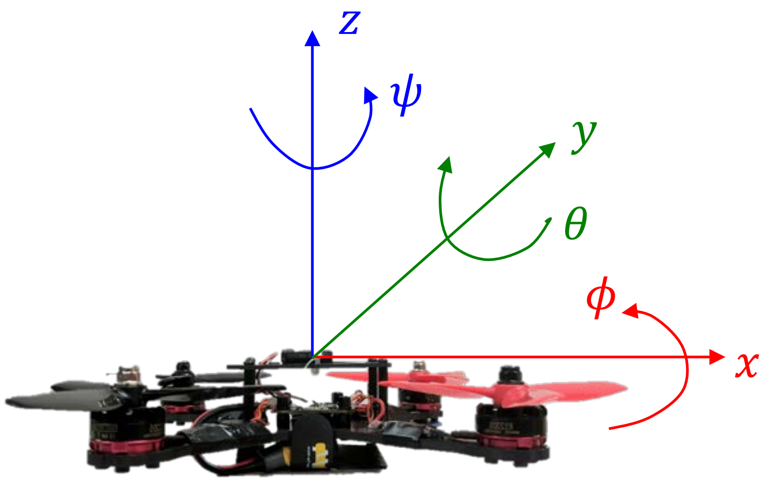
Remember as the word **"pre-fix"**

$$\mathbf{R}_2^0 = \mathbf{R}_2^1\mathbf{R}_1^0 = \mathbf{R}_{z,\psi}\,\mathbf{R}_{y,\theta}$$

Pre-multiply successive transformations about intermediate frames

Rotations using Euler Angles

Euler Angles: Intermediate rotation angles (ϕ, θ, ψ)



Not same as
Roll, Pitch and Yaw
Why?

How many combinations of this convention are there?

Advantages

- Minimal: 3 parameters 3 DOF
- Intuitive

Disadvantages

- Hard if convention not specified
- Will flip at singularity

Rotations using Rotation Matrix

$\mathbf{R} = \mathbf{R}_{z,\psi}\, \mathbf{R}_{y,\theta} \mathbf{R}_{x,\phi}$ For Z-Y-X Euler Angles

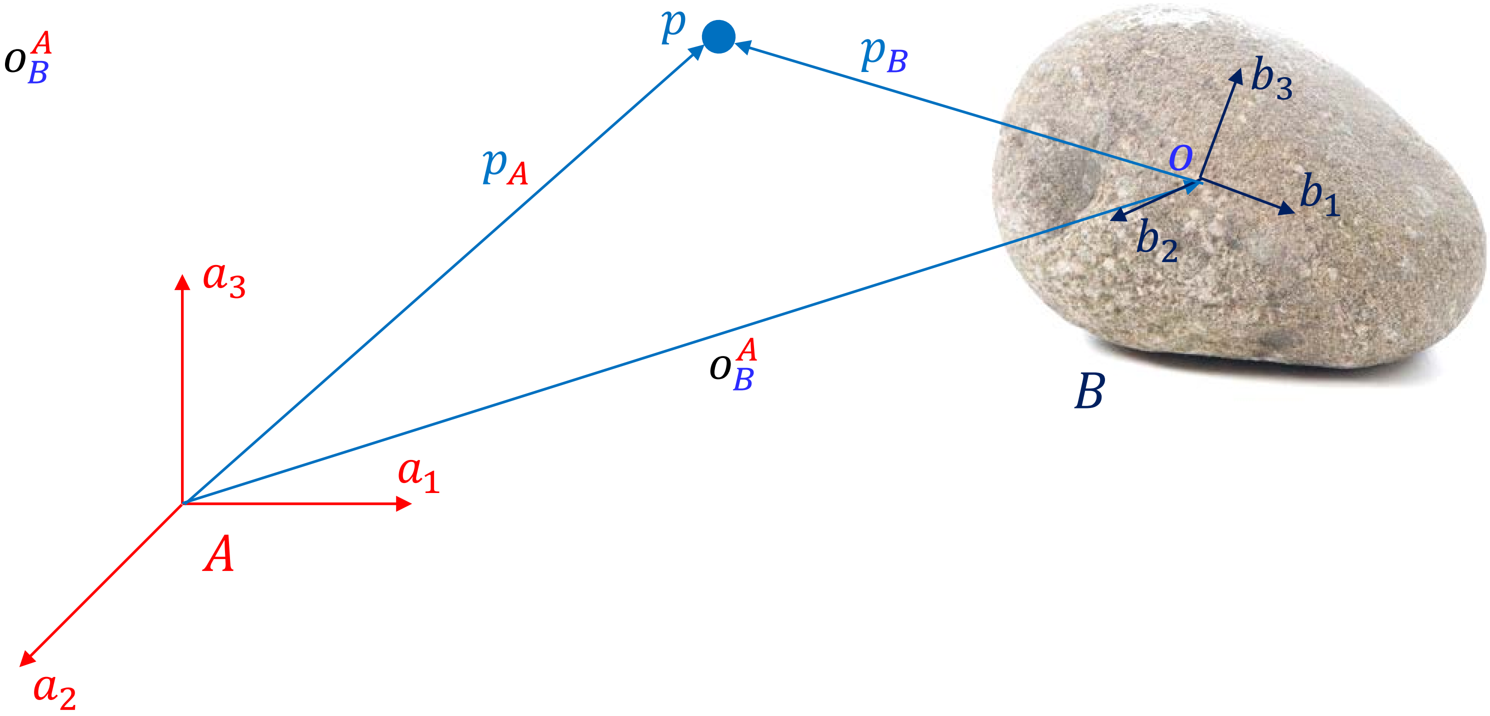
$$\mathbf{R}_{x,\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \mathbf{R}_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} R_{z,\psi} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Use `rotm2eul`, `eul2rotm` in MATLAB

What about Translation?

First rotate using a rotation matrix then shift origin using translation

$p_A = \mathbf{R}_{\mathbf{B}}^{\mathbf{A}} p_B + o_B^A$



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Prettifying using Homogeneous Transformations

$$p_A = \mathbf{R}_B^A p_B + o_B^A$$

$$P_A = \mathbf{H}_B^A P_B = \begin{bmatrix} \mathbf{R}_B^A & o_B^A \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} p_B \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_B^A p_B + o_B^A \\ 1 \end{bmatrix} = \begin{bmatrix} p_A \\ 1 \end{bmatrix}$$

Homogeneous transformation is a matrix representation of a rigid body transformation

$$\mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{T} \\ \mathbf{0} & 1 \end{bmatrix}$$

Here $\mathbf{R}$ is a 3×3 rotation matrix and $\mathbf{T}$ is a 3×1 translation vector

The **homogeneous representation** of a vector is

$P = \begin{bmatrix} p \\ 1 \end{bmatrix}$ Here p is a 3×1 vector

Multiple Transformations made easy

$$\mathbf{H}_1 = \begin{bmatrix} \mathbf{R}_1 & \mathbf{T}_1 \\ \mathbf{0} & 1 \end{bmatrix}; \mathbf{H}_2 = \begin{bmatrix} \mathbf{R}_2 & \mathbf{T}_2 \\ \mathbf{0} & 1 \end{bmatrix} \Rightarrow \mathbf{H}_{\mathbf{tot}} = \mathbf{H}_1\mathbf{H}_2 = \begin{bmatrix} \mathbf{R}_1 & \mathbf{T}_1 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}_2 & \mathbf{T}_2 \\ \mathbf{0} & 1 \end{bmatrix}$$

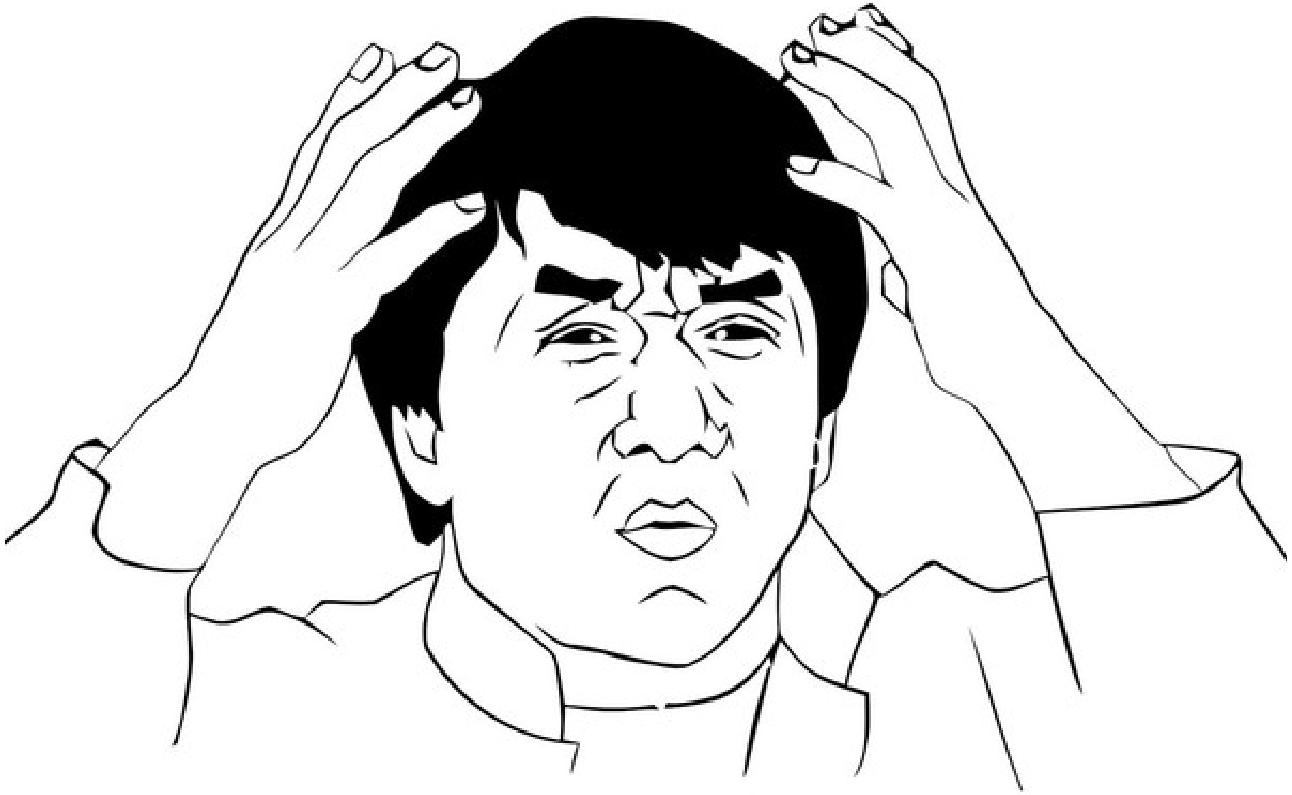
$$\mathbf{H}_{\mathbf{tot}} = \begin{bmatrix} \mathbf{R}_1\mathbf{R}_2 & \mathbf{R}_2(\mathbf{R}_1\mathbf{T}_1) + \mathbf{T}_2 \\ \mathbf{0} & 1 \end{bmatrix}$$

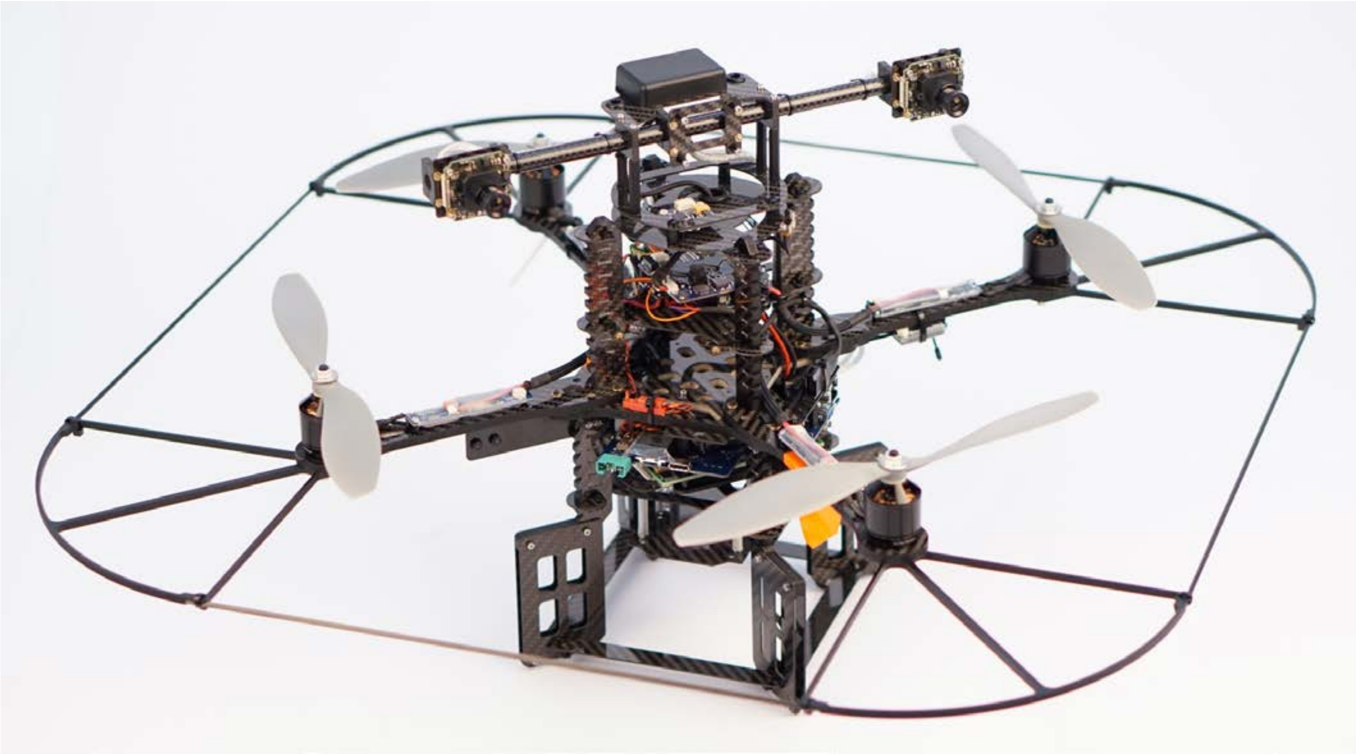
$$\mathbf{H}^{-1} = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T\mathbf{T} \\ \mathbf{0} & 1 \end{bmatrix} \neq \mathbf{H}^T$$

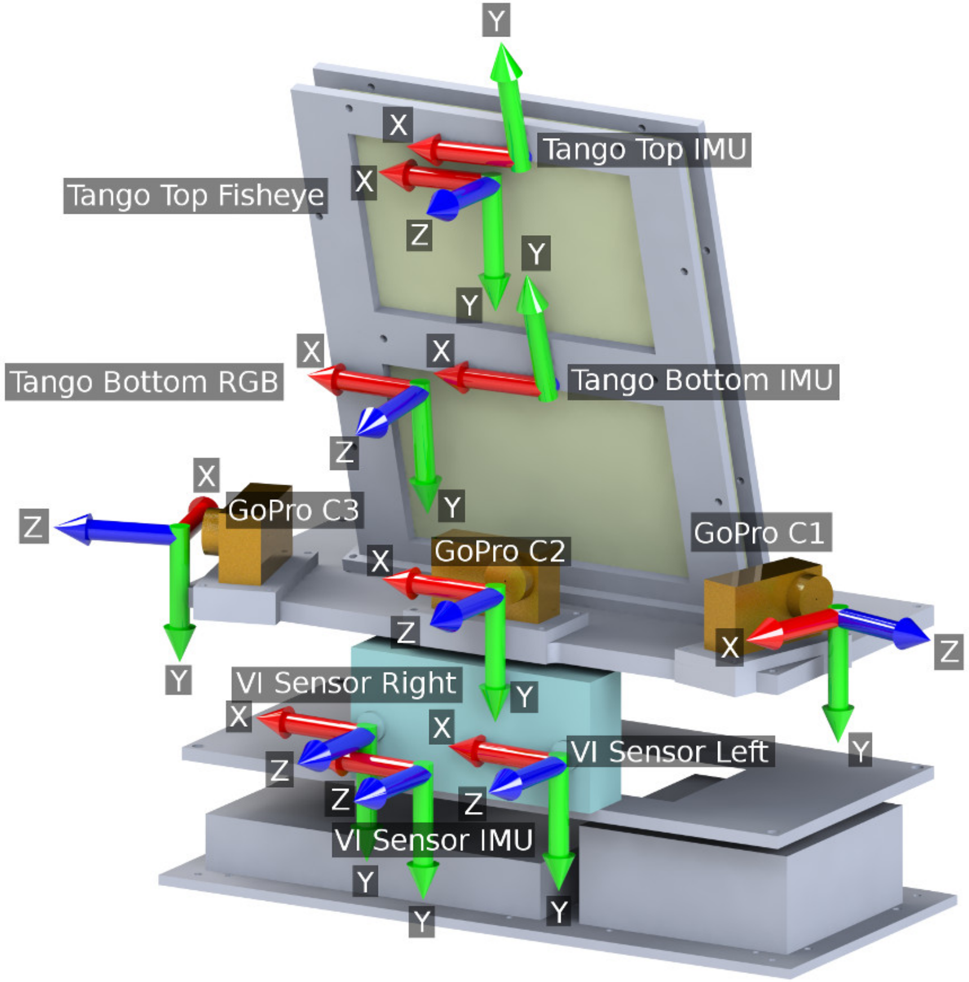
Interpretation:

- Translation of $-\mathbf{T}$ followed by rotation of $\mathbf{R}^T$ in original frame
- Rotation of $\mathbf{R}^T$ followed by translation of $-\mathbf{T}$ in current frame

How does all this go on a Quad?






Tango Top IMU
Tango Top Fisheye
Tango Bottom RGB
Tango Bottom IMU
GoPro C3
GoPro C2
GoPro C1
VI Sensor Right
VI Sensor Left
VI Sensor IMU

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} =$$