CMSC828T
Vision, Planning And Control In Aerial Robotics

VELOCITIES

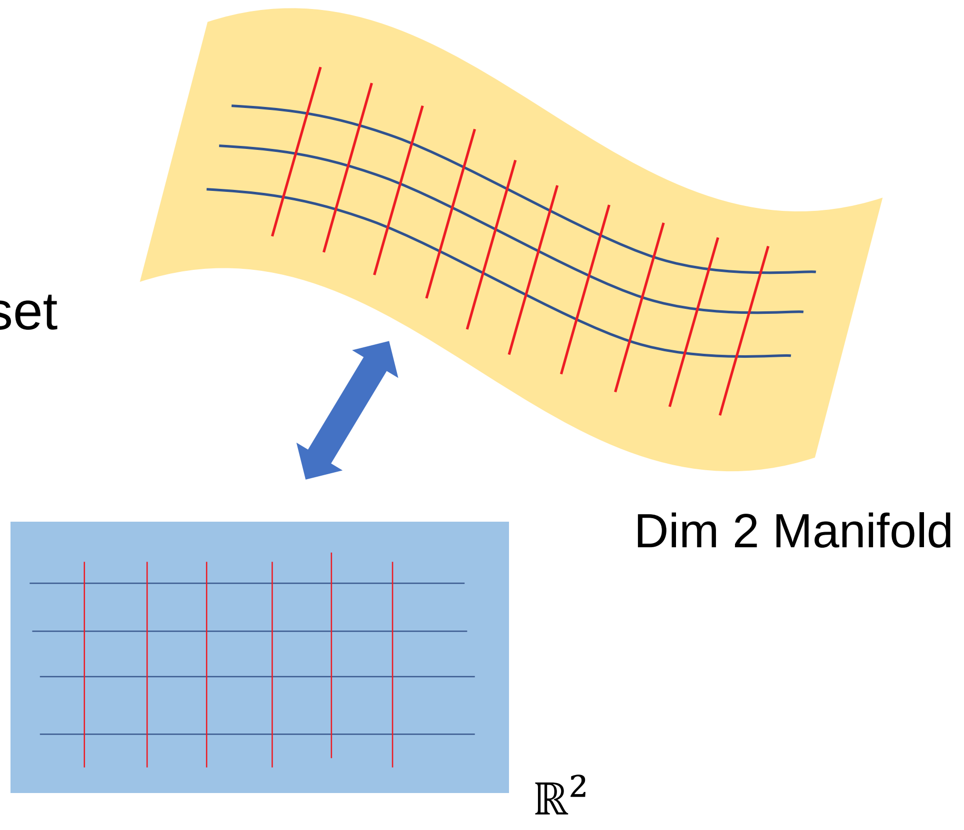
Manifold

A **manifold** of dimension n is a set M, which is **homeomorphic** to $\mathbb{R}$

A **homeomorphism** is a map $f: M \to N$ such that f and f^{-1} are both continuous

A function $f: M \to N$ is **continuous** if for each open subset $V \subset N$ the set $f^{-1}(V)$ is a open subset of M

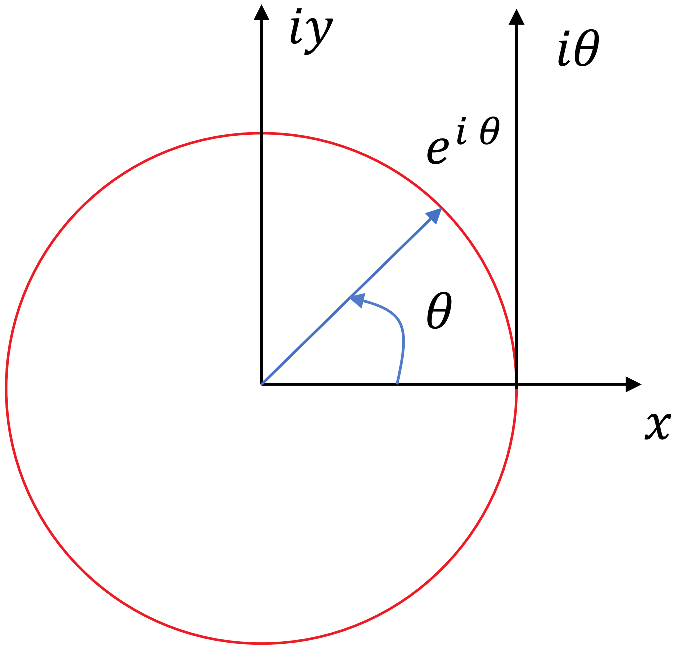
Note:
A function can be **bijective** and **continuous** without being a homeomorphism



Most of these slides are inspired by MEAM620 Slides at UPenn

Rotation on a Plane

Consider a x-y plane



$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\updownarrow$$

$$Z(\theta) = \cos\theta + i\sin\theta = e^{i\theta}$$

This can also be shown considering that

$$e^{i\theta} = 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} + \frac{i\theta^3}{3!} - \frac{\theta^4}{4!} + \cdots$$
$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$
$$= \cos\theta + i\sin\theta$$

The exponential function maps the tangent vector at 1 onto the circle of radius 1
The line can be viewed as the tangent to the circle, group $SO(2)$, at the identity element e^{i0}

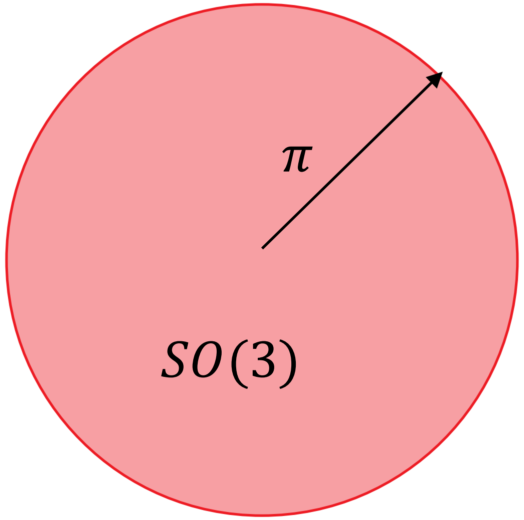
Exponential Map Onto $SO(3)$

Property 1: Exponentials of 3×3 skew symmetric matrices are rotation matrices

$$\forall \omega \in \mathbb{R}^3 \exists\, \mathbf{R} \in SO(3) \colon \mathbf{R} = \exp_{SO(3)} \widehat{\omega}$$

Property 2: The exponential map is surjective onto $SO(3)$

$$\forall\, \mathbf{R} \in SO(3)\ \exists\ \omega \in \mathbb{R}^3 \colon \widehat{\omega} = \log_{SO(3)} \mathbf{R}$$



Definition:
The set of all 3×3 skew symmetric matrices is a **Lie algebra**, denoted by $\mathfrak{so}(3)$, for the **Lie group** $SO(3)$.

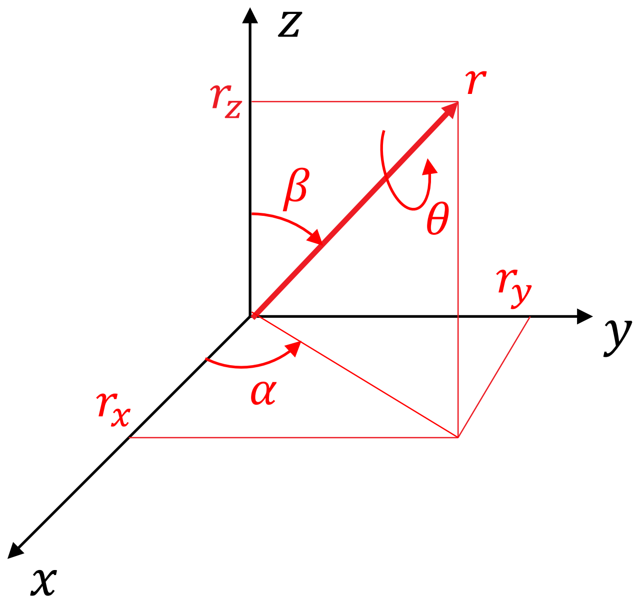
Rotation about an Axis

Called **Rodrigues' Formula**

$$\mathbf{R}_{r,\theta} = \mathbf{I}\cos\theta + rr^T(1-\cos\theta) + \hat{r}\sin\theta$$

$$\cos\theta = \frac{(\mathrm{tr}\,\mathbf{R}) - \mathbf{1}}{2}$$

$$\hat{r} = \frac{\theta}{2\sin\theta}(\mathbf{R} - \mathbf{R}^T)$$



The inverse formula of the $\cos$ already restrict the interval to $[0, \pi]$ making it one-to-one with respect to $\mathbf{R}$

Exponential of a 3×3 Skew Symmetric Matrix

The matrix exponential is a matrix function on square matrices

$$\exp \hat{\mathbf{A}} = \mathbf{I} + \hat{\mathbf{A}} + \frac{1}{2!}\hat{\mathbf{A}}^2 + \frac{1}{3!}\hat{\mathbf{A}}^3 + \frac{1}{4!}\hat{\mathbf{A}}^4 + \frac{1}{5!}\hat{\mathbf{A}}^5 + \cdots$$

Write in terms of a unit vector and magnitude $\hat{\mathbf{A}} = \hat{\mathbf{u}}\theta$

$$\exp \hat{\mathbf{u}}\theta = \mathbf{I} + \hat{\mathbf{u}}\theta + \frac{\theta^2}{2!}\hat{\mathbf{u}}^2 + \frac{\theta^3}{3!}\hat{\mathbf{u}}^3 + \frac{\theta^4}{4!}\hat{\mathbf{u}}^4 + \frac{\theta^5}{5!}\hat{\mathbf{u}}^5 + \cdots$$

We have $\hat{\mathbf{u}}^{\mathbf{4}} = -\hat{\mathbf{u}}^2$, $\hat{\mathbf{u}}^{\mathbf{5}} = \hat{\mathbf{u}}$, $\hat{\mathbf{u}}^{\mathbf{6}} = \hat{\mathbf{u}}^2$, $\hat{\mathbf{u}}^{\mathbf{7}} = -\hat{\mathbf{u}}$ **Why?**

$$\exp \hat{\mathbf{u}}\theta = \mathbf{I} + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)\hat{\mathbf{u}} + \left(\theta + \frac{\theta^2}{2!} + \frac{\theta^4}{5!} - \cdots\right)\hat{\mathbf{u}}^{\mathbf{2}}$$

$$\exp \hat{\mathbf{u}}\theta = \mathbf{I} + \hat{\mathbf{u}}\sin\theta + \hat{\mathbf{u}}^{\mathbf{2}}(1 - \cos\theta)$$

$\hat{\mathbf{u}}^{\mathbf{2}} = \mathbf{u}\mathbf{u}^{\mathbf{T}} - \mathbf{I}$ **Rodrigues' Formula! ☺**

Verify that $\exp \hat{\mathbf{u}}\theta$ is indeed a Rotation Matrix

Remember definition of a rotation matrix

$$SO(3) = \{\mathbf{R} \in \mathbb{R}^{3\times 3} | \mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}, \det \mathbf{R} = +1\}$$

$$\exp(\hat{\mathbf{u}}\theta)^{-1} = \exp(-\hat{\mathbf{u}}\boldsymbol{\theta}) = \exp(\hat{\mathbf{u}}^T\theta) = \exp(\hat{\mathbf{u}}\theta)^T$$

Now we need to show that the determinant is $+1$

$$\exp(\hat{\mathbf{u}}\theta)^T \exp(\hat{\mathbf{u}}\theta) = \mathbf{I}$$

$$\det\left(\exp(\hat{\mathbf{u}}\theta)^T \exp(\hat{\mathbf{u}}\theta)\right) = \det\left(\exp(\hat{\mathbf{u}}\theta)^T\right) \det(\exp(\hat{\mathbf{u}}\theta)) = 1$$

$\det A = \det A^T$ so the determinant has to be 1

This means that any exponential of $\hat{\mathbf{u}}\theta$ with $\| \mathbf{u} \| = 1$ is a rotation matrix

The opposite is also true and can be shown using a constructive proof

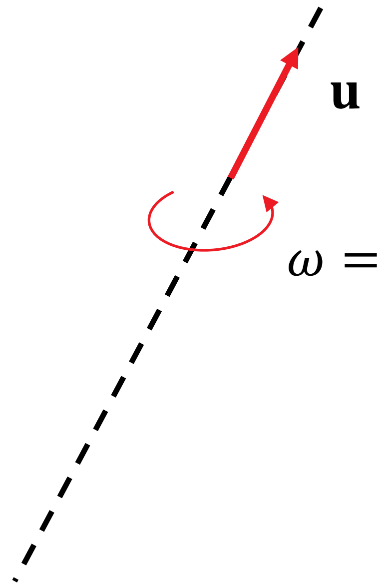
Physical Interpretation

Consider a rotation of angle $d\theta$ around axis $\mathbf{u}$

$$\mathbf{u}d\theta = \omega\, dt$$

$$\exp(\hat{\mathbf{u}}d\theta) = \mathbf{I} + \hat{\mathbf{u}}\sin d\theta + \hat{\mathbf{u}}^2(1 - \cos d\theta)$$

$$\exp(\hat{\boldsymbol{\omega}}dt) = \mathbf{I} + \frac{\omega}{\|\,\omega\,\|}\sin(\omega dt) + \frac{\omega}{\|\,\omega\,\|^2}\,(1 - \cos(\omega dt))$$



$$\omega = \mathbf{u}\frac{d\theta}{dt}$$

Derivative of a Rotation Matrix

Derivative of a rotation matrix

$$\mathbf{R}(\mathrm{t})^{\mathrm{T}}\mathbf{R}(t) = \mathbf{I}$$

Differentiate by using product rule

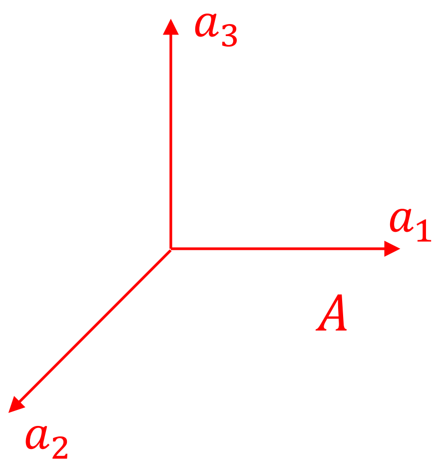
$$\mathbf{R}(\mathrm{t})^{\mathrm{T}}\dot{\mathbf{R}}(t) + \dot{\mathbf{R}}(t)^{T}\mathbf{R}(t) = \mathbf{0}$$

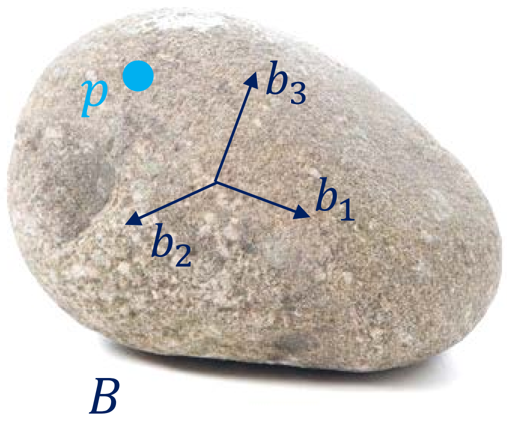
$$\mathbf{S}(\mathrm{t})^{\mathrm{T}} + \mathbf{S}(t) = \mathbf{0}$$

For any **skew symmetric matrix**, $\mathbf{A}^{T} = -\mathbf{A}$,hence $\mathbf{S}$ is skew symmetric matrix

$$a \times b = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \mathbf{S}(a)b$$

Angular Velocity





Angular Velocity

$q(t) = \mathbf{R}(t)p$ Given angular velocities in body fixed frame **B** we want to find angular velocities in world fixed frame **A**

$\dot{q} = \dot{\mathbf{R}}p$

$\dot{q}(t) = \dot{\mathbf{R}}(t)p = S(\omega(t))\mathbf{R}(t)p$ From previous slides

If $\omega(t)$ is the angular velocity with respect to the reference frame, with rotation $\mathbf{R}(t)$ at time t . From mechanics we know that,

$\dot{q}(t) = \omega(t) \times \mathbf{R}(t)\, p$

We know that the Skew Symmetrix matrix represents a cross product, hence,

$\dot{q}(t) = \omega(t) \times \mathbf{R}(t)p = S(\omega(t))\mathbf{R}(t)p$

Velocity of a Point

$$p_A = \mathbf{R}_B^A p_B + o_B^A$$

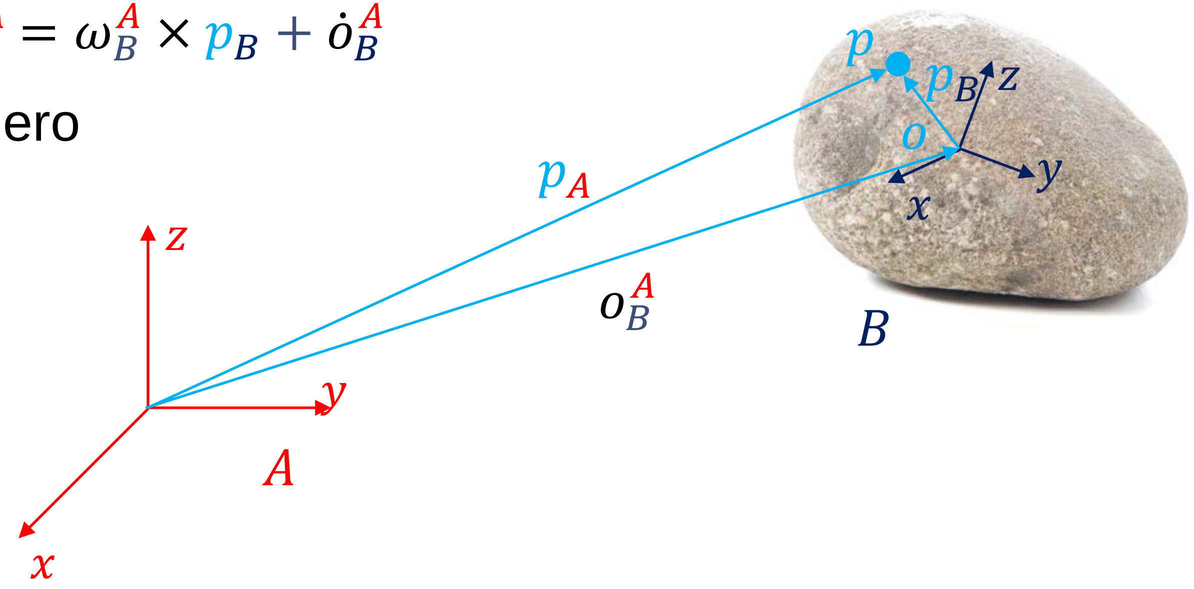
$$\dot{p}_A = \dot{\mathbf{R}}_B^A p_B + \mathbf{R}_B^A \dot{p}_B + \dot{o}_B^A \; = S(\omega_B^A)\mathbf{R}_B^A p_B + \mathbf{R}_B^A \dot{p}_B + \dot{o}_B^A = \omega_B^A \times p_B + \mathbf{R}_B^A \dot{p}_B + \dot{o}_B^A$$

Assume p is fixed in Frame B.

$$\dot{p}_A = \dot{\mathbf{R}}_B^A p_B + \mathbf{R}_B^A \dot{p}_B + \dot{o}_B^A \; = S(\omega_B^A)\mathbf{R}_B^A p_B + \dot{o}_B^A = \omega_B^A \times p_B + \dot{o}_B^A$$

If rigid body only rotates, i.e., translation is zero

$$\dot{p}_A = \omega_B^A \times p_B = S(\omega_B^A)\mathbf{R}_B^A p_B = \dot{\mathbf{R}}_B^A p_B$$



Velocity of a Point

$$\dot{p}_A = \omega_B^A \times p_B = S(\omega_B^A)\mathbf{R}_B^A p_B = \dot{\mathbf{R}}_B^A p_B$$

Multiply both sides by ${\mathbf{R}_B^A}^T$

$${\mathbf{R}_B^A}^T \dot{p}_A = {\mathbf{R}_B^A}^T \dot{\mathbf{R}}_B^A p_B$$

Velocity in body fixed frame

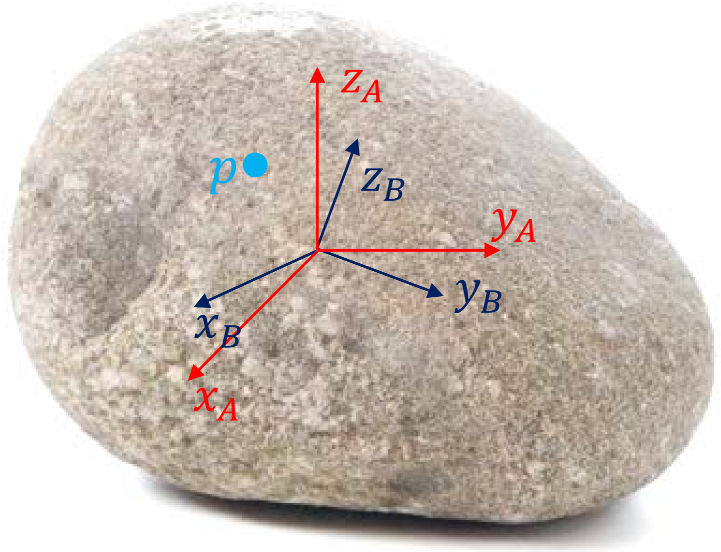
Encodes angular velocity in body fixed frame

Rewrite p_B in the inertial frame as $p_B = {\mathbf{R}_B^A}^T\ p_B^A$

$$\dot{p}_A = \dot{\mathbf{R}}_B^A {\mathbf{R}_B^A}^T p_B^A$$

Velocity in inertial frame

Encodes angular velocity in inertial frame



Angular Velocity

$\mathbf{R}^T \dot{q}(t) = \mathbf{R}^{\mathrm{T}} \dot{\mathbf{R}}(t) p$

Velocity in body fixed frame

Encodes angular velocity in body fixed frame

$\dot{q} = \dot{\mathbf{R}} \mathbf{R}^{\mathrm{T}} q$

Velocity in inertial frame

Encodes angular velocity in inertial frame

Skew Symmetric

$\widehat{\omega}^b = \mathbf{R}^T \dot{\mathbf{R}}$

Angular velocity in body fixed frame w.r.t inertial frame in body fixed frame

$\widehat{\omega}^s = \dot{\mathbf{R}} \mathbf{R}^T$

Angular velocity in body fixed frame w.r.t. inertial frame in inertial frame

Angular velocity in body fixed Frame

$\mathbf{R} = \mathbf{R} \widehat{\omega}^b \qquad \mathbf{R}(t + \delta t) \sim \mathbf{R}(t) + \delta t \mathbf{R}(t) \widehat{\omega}^b$

Angular velocity in inertial Frame

$\mathbf{R} = \widehat{\omega}^s \mathbf{R} \qquad \mathbf{R}(t + \delta t) \sim \mathbf{R}(t) + \delta t \widehat{\omega}^s \mathbf{R}(t)$

[Murray *et al*, 1994]

Angular Velocities and Euler Angles

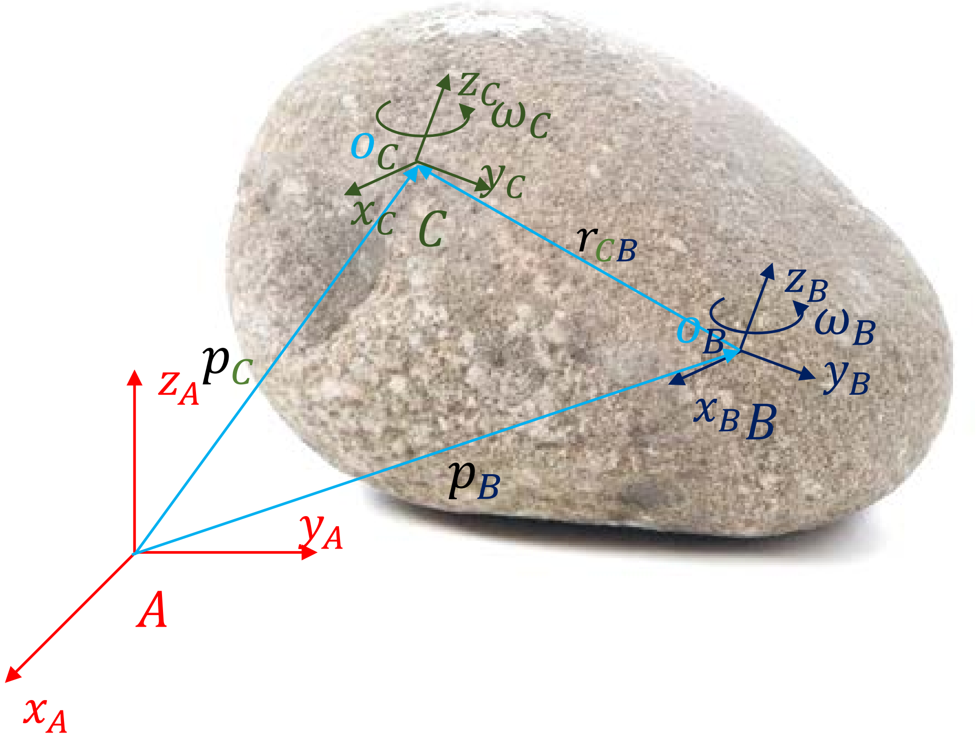
Assume ZXY Euler Angle Parametrization

For the quadrotor you will have to determine the relation between the angular velocities and the orientation to determine the attitude dynamic model

$$\omega = \mathbf{T}\dot{\phi}_e = \begin{bmatrix} \cos\theta & 0 & -\cos\phi\sin\theta \\ 0 & 1 & \sin\phi \\ \sin\theta & 0 & \cos\phi\cos\theta \end{bmatrix} \dot{\phi}_e$$

Transforming Velocities

Consider two frames on the same rigid body



$$r_{CB} = \mathbf{R}_C r_{CB}^C$$

$$\omega_C = \omega_B$$

$$\dot{p}_B = \dot{p}_C + S(\omega_C) r_{CB} = \dot{p}_C - S(r_{CB})\omega_C$$

$$\begin{bmatrix} \dot{p}_B \\ \omega_B \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -S(r_{CB}) \\ 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{p}_C \\ \omega_C \end{bmatrix}$$

$$\dot{p}_C = \mathbf{R}_C \dot{p}_C^C \qquad \omega_C = \mathbf{R}_C \omega_C^C$$

$$\dot{p}_B = \mathbf{R}_B \dot{p}_B^B = \mathbf{R}_C \mathbf{R}_B^C \dot{p}_B^B \qquad \omega_B = \mathbf{R}_B \omega_B^B = \mathbf{R}_C \mathbf{R}_B^C \omega_B^B$$

$$\mathbf{R} S(\omega) \mathbf{R}^T = S(\mathbf{R}\omega)$$

$$\mathbf{R}_C \mathbf{R}_B^C \dot{p}_B^B = \mathbf{R}_C \dot{p}_C^C - \mathbf{R}_C S(r_{CB}^C) \mathbf{R}_C^T \mathbf{R}_C \omega_C^C$$

$$\mathbf{R}_C \mathbf{R}_B^C \omega_B^B = \mathbf{R}_C \omega_C^C$$

$$\begin{bmatrix} \dot{p}_B^B \\ \omega_B^B \end{bmatrix} = \begin{bmatrix} \mathbf{R}_C^B & -\mathbf{R}_C^B S(r_{CB}^C) \\ 0 & \mathbf{R}_C^B \end{bmatrix} \begin{bmatrix} \dot{p}_C^C \\ \omega_C^C \end{bmatrix}$$

Homogeneous Transformations

Homogeneous transformation is a matrix representation of a rigid body transformation

$$\mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{T} \\ \mathbf{0} & 1 \end{bmatrix}$$

Here $\mathbf{R}$ is a 3×3 rotation matrix and $\mathbf{T}$ is a 3×1 translation vector

The **homogeneous representation** of a vector is

$P = \begin{bmatrix} p \\ 1 \end{bmatrix}$ Here p is a 3×1 vector

$$SE(3) = \{\mathbf{H} \in \mathbb{R}^{4\times4} | \mathbf{R} \in SO(3), \mathbf{T} \in \mathbb{R}^{3\times1}\}$$

Derivative of Homogeneous Transformation

For a rotation matrix we know that

$$S(t) = \dot{\mathbf{R}}(t)\mathbf{R}(t)^T$$

Similarly for a homogeneous transform we have

$$\dot{\mathbf{H}}(t) = S_h(t)\mathbf{H}(t)$$

$$S_h(t) = \begin{bmatrix} \dot{\mathbf{R}}(t)\mathbf{R}(t)^T & \dot{\mathbf{T}}(t) - \dot{\mathbf{R}}(t)\mathbf{R}(t)^T\mathbf{T}(t) \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \dot{\mathbf{H}}(t)\dot{\mathbf{H}}(t)^{-1} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} = \hat{\boldsymbol{\xi}}$$

This is called twist

Exponential Map onto $SE(3)$

Property 1: Exponentials of 4×4 matrix of this type are homogeneous transforms $\mathbf{H}$

$$\hat{\boldsymbol{\xi}} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix}$$

$$\forall \boldsymbol{\xi} \in \mathbb{R}^{6\times6} \exists \mathbf{H} \in SE(3): \mathbf{H} = \exp_{\mathrm{SE}(3)} \hat{\boldsymbol{\xi}}$$

Property 2: The exponential map is surjective onto *SE*(3).

$$\forall \mathbf{H} \in SE(3) \exists \boldsymbol{\xi} \in \mathbb{R}^{6\times6}: \hat{\boldsymbol{\xi}} = \log_{\mathrm{SE}(3)} \mathbf{H}$$

Definition:
The set of all 4×4 matrices of that type is a **Lie algebra**, denoted by $\mathfrak{se}(3)$, for the Lie group $SE(3)$

Derivative of Homogeneous Transformation

Consider a constant vector p with

$Q(t) = \mathbf{H}_{\mathbf{B}}^{\mathbf{A}}(t)P$

$\dot{Q}(t) = \dot{\mathbf{H}}_{\mathbf{B}}^{\mathbf{A}}(t)P = S_h(t)\mathbf{H}(t)P$

P is expressed in the body frame

$H_{\boldsymbol{B}}^{\boldsymbol{A}}(t)$ transforms the vector into inertial frame

$\mathbf{H}_{\boldsymbol{B}}^{\boldsymbol{A}}(t)^{-1}\dot{Q}(t) = \boxed{\mathbf{H}_{\mathbf{B}}^{\mathbf{A}}(\mathbf{t})^{-1}\dot{\mathbf{H}}_{\mathbf{B}}^{\mathbf{A}}(t)}P$

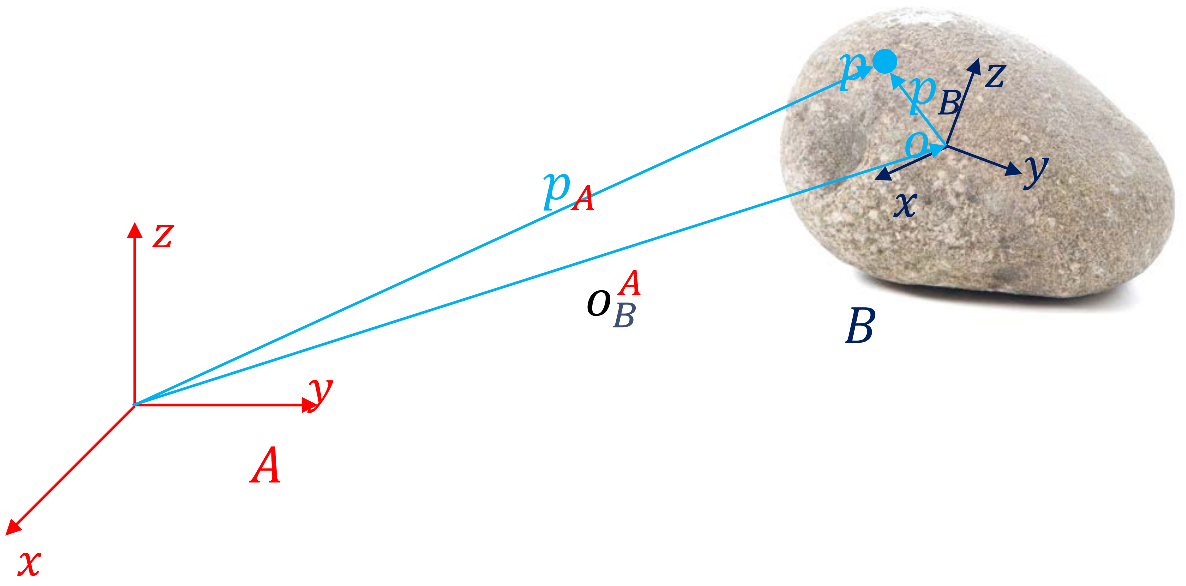
Velocity in body fixed frame

Encodes velocities in body fixed frame $\boldsymbol{S_h(t)}$

$\dot{Q}(t) = \dot{\mathbf{H}}_{\mathbf{B}}^{\mathbf{A}}(t)P = \boxed{\dot{\mathbf{H}}_{\mathbf{B}}^{\mathbf{A}}(t)\mathbf{H}_{\mathbf{B}}^{\mathbf{A}}(t)^{-1}}Q(t)$

Velocity in inertial frame

Encodes velocities in inertial frame



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Velocity of a Point

$$S_h(t) = \begin{bmatrix} \dot{\mathbf{R}}(t)\mathbf{R}(t)^T & \dot{\mathbf{T}}(t) - \dot{\mathbf{R}}(t)\mathbf{R}(t)^T\mathbf{T}(t) \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \dot{\mathbf{H}}(t)\dot{\mathbf{H}}(t)^{-1} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} = \hat{\boldsymbol{\xi}}$$

$$p_A = \mathbf{R}_B^A p_B + o_B^A$$

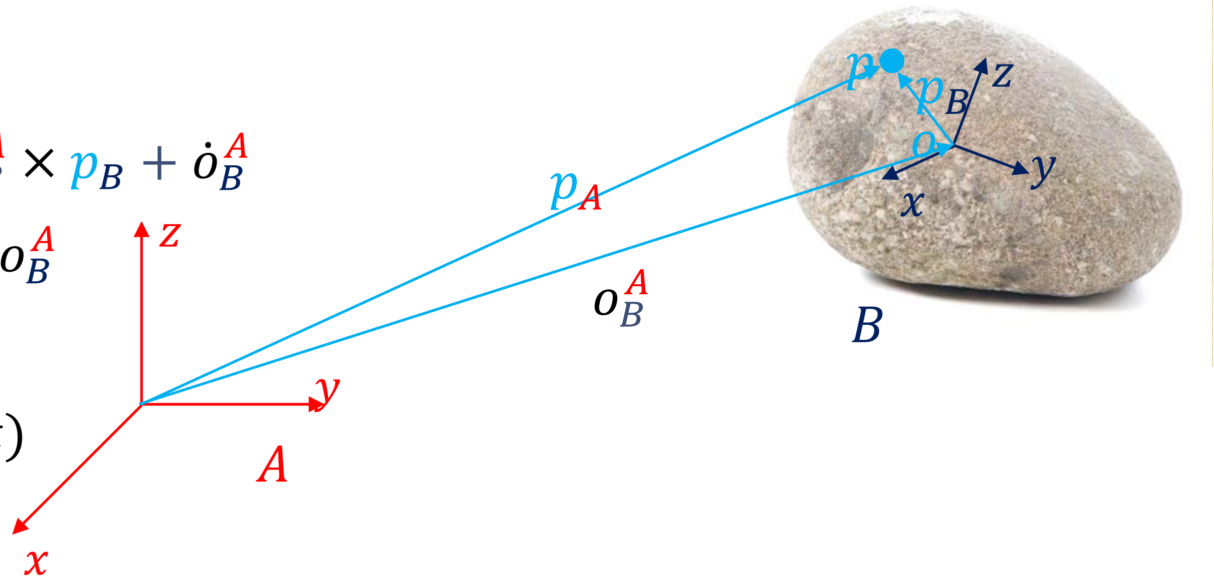
$$\dot{p}_A = \dot{\mathbf{R}}_B^A p_B + \mathbf{R}_B^A \dot{p}_B + \dot{o}_B^A = S(\omega_B^A)\mathbf{R}_B^A p_B + \mathbf{R}_B^A \dot{p}_B + \dot{o}_B^A = \omega_B^A \times p_B + \mathbf{R}_B^A \dot{p}_B + \dot{o}_B^A$$

Assume p is fixed in Frame B.

$$\dot{p}_A = \dot{\mathbf{R}}_B^A p_B + \mathbf{R}_B^A \dot{p}_B + \dot{o}_B^A = S(\omega_B^A)\mathbf{R}_B^A p_B + \dot{o}_B^A = \omega_B^A \times p_B + \dot{o}_B^A$$

$$= S(\omega_B^A)(p_A - o_B^A) + \dot{o}_B^A = S(\omega_B^A)p_A + \dot{o}_B^A - S(\omega_B^A)\, o_B^A$$

Same as $\dot{\mathbf{H}}(t)\mathbf{H}(t)^{-1}Q(t)$



Summary

	Rotation	Pose
Matrix	$\boldsymbol{R} \in \mathbb{R}^{3\times3} \mid \mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}, \det \mathbf{R} = +1$	$\mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{T}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 \end{bmatrix}$
Vector	$q = \mathbf{R}p$	$Q = \mathbf{H}P$
Body frame velocities	$\widehat{\omega}^b = \mathbf{R}^T\dot{\mathbf{R}}$	$\hat{\boldsymbol{\xi}}^{\boldsymbol{b}} = \boldsymbol{H}^{-1}\dot{\boldsymbol{H}}$
Inertial frame velocities	$\widehat{\omega}^s = \dot{\mathbf{R}}\mathbf{R}^T$	$\hat{\boldsymbol{\xi}}^{\boldsymbol{s}} = \dot{\boldsymbol{H}}\boldsymbol{H}^{-1}$

Moving velocities between different **moving** frames

This is often called **adjoint**

$$\begin{bmatrix} \dot{p}_B^B \\ \omega_B^B \end{bmatrix} = \begin{bmatrix} \mathbf{R}_C^B & -\mathbf{R}_C^B S(r_{CB}^C) \\ 0 & \mathbf{R}_C^B \end{bmatrix} \begin{bmatrix} \dot{p}_C^C \\ \omega_C^C \end{bmatrix}$$